

Fractional Calculus: Mathematical Challenges and New Directions

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- Fractional calculus is the generalization of integration and differentiation operators $\frac{d^n f}{dt^n}$ from integer order $n \in \mathbb{N}$ to an arbitrary non-integer order $\alpha \in \mathbb{R}$ or $\mathbb{C}$.
- The formal origin of the field dates back to 1695, when L'Hôpital inquired of Leibniz about the meaning of the half-derivative $\frac{d^{1/2} f}{dt^{1/2}}$.
- Non-integer order operators have become fundamental modeling tools in mathematics, physics, and engineering to describe memory-dependent and non-local systems.
- Despite major advances in the domain, several elementary and basic properties still require rigorous mathematical proofs.

Riemann-Liouville fractional integral

Liouville, Riemann, and the Integral Approach

Repeated Integration Pattern: Let $f \in L^1(t_0, T)$.

- First integral: $I^1 f(t) = \int_{t_0}^t f(s) ds$
- Second integral: $I^2 f(t) = \int_{t_0}^t I^1 f(s) ds = \int_{t_0}^t (t-s)f(s) ds$
- n -th integral ($n \in \mathbb{N}$):

$$\begin{aligned} I^n f(t) &= \int_{t_0}^t I^{n-1} f(s) ds \\ &= \frac{1}{(n-1)!} \int_{t_0}^t (t-s)^{n-1} f(s) ds \end{aligned} \tag{1}$$

$$= (k_n * f)(t) \tag{2}$$

where $k_n(t) = \frac{t^{n-1}}{(n-1)!}$.

Riemann-Liouville Fractional Integral

Gamma Function and Riemann-Liouville Kernel

- $(n-1)! = \Gamma(n)$ for all $n \in \mathbb{N}$
- $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$ for all $\alpha > 0$
- Riemann-Liouville kernel: $k_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$

Definition (Riemann-Liouville Fractional Integral)

The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function f at time $t \in (t_0, T]$ is defined by:

$${}^{\text{RL}}_{t_0} I^\alpha f(t) = (k_\alpha * f)(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^t (t-s)^{\alpha-1} f(s) ds \quad (3)$$

whenever the function $s \mapsto (t-s)^{\alpha-1} f(s)$ is Lebesgue integrable on $[t_0, t]$.

Operator properties of ${}^{\text{RL}}I_{t_0}^\alpha$ on $L^1(t_0, T)$

Proposition

Let $\alpha, \beta > 0$.

1. **Well-definedness:** If $f \in L^1(t_0, T)$, then ${}^{\text{RL}}I_{t_0}^\alpha f$ is well-defined almost everywhere on $[t_0, T]$, and

$${}^{\text{RL}}I_{t_0}^\alpha f \in L^1(t_0, T)$$

2. **Bounded operator:** The map

$${}^{\text{RL}}I_{t_0}^\alpha : L^1(t_0, T) \longrightarrow L^1(t_0, T)$$

is bounded.

3. **Semigroup property:**

$${}^{\text{RL}}I_{t_0}^\alpha \circ {}^{\text{RL}}I_{t_0}^\beta = {}^{\text{RL}}I_{t_0}^{\alpha+\beta} \quad (4)$$

4. **Injectivity:** ${}^{\text{RL}}I_{t_0}^\alpha$ is injective.

Mapping and Regularity Properties of ${}^{\text{RL}}I_{t_0}^\alpha$ on $L^\infty(t_0, T)$

Proposition

Let $\alpha > 0$.

1. **Continuity:** If $f \in L^\infty(t_0, T)$, then ${}^{\text{RL}}I_{t_0}^\alpha f$ is continuous on $(t_0, T]$ and admits a continuous extension to t_0 with

$${}^{\text{RL}}I_{t_0}^\alpha f(t_0) = 0$$

2. **Regularization effect:** The operator

$${}^{\text{RL}}I_{t_0}^\alpha : L^\infty(t_0, T) \longrightarrow \mathcal{C}([t_0, T])$$

is bounded and satisfies the semigroup property on $[t_0, T]$.

3. **Hölder continuity:** If $\alpha \in (0, 1)$ and $f \in L^\infty(t_0, T)$, then

$${}^{\text{RL}}I_{t_0}^\alpha f \in \mathcal{H}^\alpha([t_0, T])$$

where $\mathcal{H}^\alpha([t_0, T])$ is the space of Hölder-continuous functions of order α .

Operator properties of ${}^{\text{RL}}_{t_0}I^\alpha$ on $\mathcal{C}([t_0, T])$

Proposition

Let $\alpha > 0$.

1. **Continuity preservation:** If $f \in \mathcal{C}([t_0, T])$, then

$${}^{\text{RL}}_{t_0}I^\alpha f \in \mathcal{C}([t_0, T])$$

2. **Refined Hölder regularity:** If $\alpha \in (0, 1)$ and $f \in \mathcal{C}([t_0, T])$, then

$${}^{\text{RL}}_{t_0}I^\alpha [f - f(t_0)] \in \mathcal{H}_0^\alpha([t_0, T])$$

where $\mathcal{H}_0^\alpha([t_0, T])$ is the closed subspace of $\mathcal{H}^\alpha([t_0, T])$ with functions u satisfying:

$$\sup_{0 \leq s < t \leq T, t-s \leq \epsilon} \frac{|u(t) - u(s)|}{(t-s)^\alpha} \xrightarrow{\epsilon \rightarrow 0} 0$$

Operator properties of ${}^{\text{RL}}_{t_0}I^\alpha$ on $\mathcal{H}^\beta([t_0, T])$

Proposition

Let $\alpha > 0$ and $\beta \in (0, 1)$.

1. **Regularity enhancement:** If $\alpha + \beta < 1$ and $f \in \mathcal{H}^\beta([t_0, T])$, then

$${}^{\text{RL}}_{t_0}I^\alpha f \in \mathcal{H}^{\alpha+\beta}([t_0, T])$$

2. **Hardy–Littlewood property:** If $\alpha + \beta > 1$ and $f \in \mathcal{H}^\beta([t_0, T])$, then

$${}^{\text{RL}}_{t_0}I^\alpha f \in \mathcal{C}^{1, \alpha+\beta-1}([t_0, T])$$

3. **Boundedness:** The operator ${}^{\text{RL}}_{t_0}I^\alpha$ is continuous (bounded) between these spaces:

$$\|{}^{\text{RL}}_{t_0}I^\alpha f\|_{\mathcal{H}^{\alpha+\beta}} \leq C \|f\|_{\mathcal{H}^\beta}, \quad \text{for } \alpha + \beta < 1$$

Ref.: Samko, S. G., Kilbas, A. A., Marichev, O. I. (1993). Fractional Integrals and Derivatives: Theory and Applications. Gordon and Breach Science Publishers, Y-Swiss. (Chapter 3).

Conclusion and key examples

Fractional vs. Integer Order Regularization

While classical integration ($\alpha = 1$) adds a full integer order of differentiability mapping $\mathcal{C}([t_0, T]) \rightarrow \mathcal{C}^1([t_0, T])$, the fractional integral ${}_{t_0}^{\text{RL}}I^\alpha$ with $\alpha \in (0, 1)$ yields a **milder fractional smoothing effect**, regularizing inputs by order α mapping $\mathcal{C}([t_0, T]) \rightarrow \mathcal{H}_0^\alpha([t_0, T])$.

Proposition (Power functions)

Let $\alpha > 0$. For $\beta > -1$

$${}_{t_0}^{\text{RL}}I^\alpha(t - t_0)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta + \alpha + 1)}(t - t_0)^{\beta + \alpha}$$

Mittag–Leffler and fractional exponential functions

Definition

- The **Mittag–Leffler function** with parameter $\alpha > 0$ is

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}$$

- The **Mittag–Leffler function** with two parameters $\alpha, \beta > 0$ is

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + \beta)}$$

- The **fractional exponential** of order $\alpha > 0$ and parameter $\lambda \neq 0$ is

$$e_{\alpha,\lambda}(t) = E_{\alpha}(\lambda(t - t_0)^{\alpha})$$

- **Fundamental Formula** : For $\lambda \neq 0$

$${}^{RL}_{t_0}I^{\alpha} e_{\alpha,\lambda} = \frac{1}{\lambda}(e_{\alpha,\lambda} - 1)$$

Two Caputo operators in the literature

Riemann-Liouville Fractional Derivative

Construction for $\alpha \in (0, 1)$

Let $f, g \in L^1(t_0, T)$.

$$\begin{aligned} {}^{\text{RL}}I_{t_0}^{\alpha} f = g &\Leftrightarrow {}^{\text{RL}}I_{t_0}^{1-\alpha} {}^{\text{RL}}I_{t_0}^{\alpha} f = {}^{\text{RL}}I_{t_0}^{1-\alpha} g \\ &\Leftrightarrow I^1 f = {}^{\text{RL}}I_{t_0}^{1-\alpha} g \\ &\Leftrightarrow f = \frac{d}{dt} {}^{\text{RL}}I_{t_0}^{1-\alpha} g \end{aligned}$$

General Definition

For $\alpha > 0$, let $n = \lceil \alpha \rceil$ (ceiling of α). The **Riemann-Liouville fractional derivative** is:

$${}^{\text{RL}}D_{t_0}^{\alpha} f(t) = \frac{d^n}{dt^n} {}^{\text{RL}}I_{t_0}^{n-\alpha} f(t)$$

Key Property: ${}^{\text{RL}}D_{t_0}^{\alpha} C = \frac{C(t - t_0)^{-\alpha}}{\Gamma(1 - \alpha)} \neq 0$

Caputo fractional derivative operators ($0 < \alpha < 1$)

First Caputo Operator: ${}^C_{t_0}\mathcal{D}^\alpha$

$$\begin{aligned} {}^C_{t_0}\mathcal{D}^\alpha f(t) &= {}^{\text{RL}}_{t_0}I^{1-\alpha} f'(t) \\ &= \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^t (t-s)^{-\alpha} f'(s) ds \end{aligned}$$

Second Caputo Operator: ${}^C_{t_0}D^\alpha$

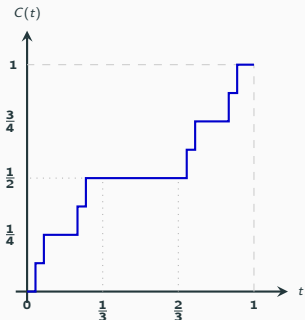
$$\begin{aligned} {}^C_{t_0}D^\alpha f(t) &= {}^{\text{RL}}_{t_0}D^\alpha [f(t) - f(t_0)] \\ &= \frac{d}{dt} {}^{\text{RL}}_{t_0}I^{1-\alpha} [f(t) - f(t_0)] \end{aligned}$$

Important

- The operators ${}^C_{t_0}\mathcal{D}^\alpha$ and ${}^C_{t_0}D^\alpha$ are generally distinct (e.g., for Cantor and Weierstrass functions).
- The second definition offers a significant conceptual advantage:
 - It does not strictly require f' to exist;
 - It depends solely on the regularity of the fractional integral;
 - It is well suited for continuous functions.

Ref.: Ross, B., Samko, S. G., & Love, E. R. (1994). Functions that have no first order derivative might have fractional derivatives of all orders less than one.

Example 1: The Cantor Ternary Function



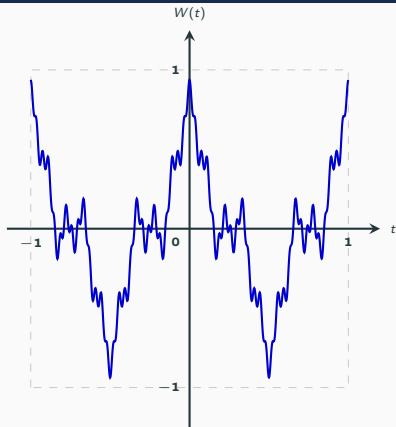
Fundamental Properties

- Continuous, non-decreasing, but **not** absolutely continuous on $[0, 1]$ ($C(0) = 0$, $C(1) = 1$).
- $C' = 0$ a.e. on $[0, 1] \setminus \mathcal{C}$ ($\lambda(\mathcal{C}) = 0$).
- Hölder continuous with exponent $\beta^c = \frac{\ln 2}{\ln 3}$.

Caputo Derivatives ($\alpha \in (0, 1)$)

- **First Caputo Operator:** ${}^c\mathcal{D}_0^\alpha C = 0$
- **Second Caputo Operator:**
 $\forall \alpha < \beta^c, \quad {}^cD_0^\alpha C \in \mathcal{H}^{\beta^c - \alpha}([0, 1]) \quad \text{and} \quad {}^cD_0^\alpha C \neq 0$

Example 2: The Weierstrass Function



Fundamental Properties

- Continuous everywhere, but **nowhere differentiable** on $\mathbb{R}$
($W(t) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi t)$).
- W' does not exist anywhere in the classical or a.e. sense.
- Hölder continuous on $[-1, 1]$ with exponent $\beta^w = \frac{\ln(1/a)}{\ln b}$ (for $ab > 1$).

Caputo Derivatives ($\alpha \in (0, 1)$)

- **First Caputo Operator:** ${}^c D_{-1}^{\alpha} W$ is **undefined** (since W' does not exist anywhere).
- **Second Caputo Operator:** $\forall \alpha < \beta^w$, ${}^c D_{-1}^{\alpha} W \in \mathcal{H}^{\beta^w - \alpha}([-1, 1])$

When do the two Caputo operators coincide? Let $\alpha \in (0, 1)$.

- If $f \in \mathcal{C}^1([t_0, T])$ (continuously differentiable):

$${}^C_{t_0}D^\alpha f = {}^C_{t_0}\mathcal{D}^\alpha f \in \mathcal{C}([t_0, T])$$

- If $f \in \mathcal{AC}([t_0, T])$ (absolutely continuous):

$${}^C_{t_0}D^\alpha f = {}^C_{t_0}\mathcal{D}^\alpha f \in L^1(t_0, T)$$

Fundamental Formulas

- **Power functions:** $\forall \beta > 0$,

$${}^C_{t_0}D^\alpha (t - t_0)^\beta = \frac{\Gamma(\beta + 1)}{\Gamma(\beta - \alpha + 1)} (t - t_0)^{\beta - \alpha}$$

- **Fractional exponentials:** $\forall \lambda \neq 0$,

$${}^C_{t_0}D^\alpha e_{\alpha, \lambda} = \lambda e_{\alpha, \lambda}$$

(Eigenfunction property, analogous to $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$)

Classical Properties: What Survives?

Linearity Holds

For all $a, b \in \mathbb{R}$,

$${}_C^C D^\alpha (af + bg) = a {}_C^C D^\alpha f + b {}_C^C D^\alpha g$$

Product Rule Fails

$${}_C^C D^\alpha (fg) \neq f {}_C^C D^\alpha g + g {}_C^C D^\alpha f$$

Chain Rule Fails

$${}_C^C D^\alpha (f \circ g) \neq f'(g) {}_C^C D^\alpha g$$

Quotient Rule Fails

$${}_C^C D^\alpha \left(\frac{f}{g} \right) \neq \frac{g {}_C^C D^\alpha f - f {}_C^C D^\alpha g}{g^2}$$

Monotonicity Fails

The sign of ${}_C^C D^\alpha f$ **does not** determines classical monotonicity.

Loss of Monotonicity Counterexample

Test Function: $X_A(t) = \frac{A}{\Gamma(\alpha+1)} t^\alpha + \sin(t)$ for $\alpha \in (0, 1)$.

1. Caputo Derivative:

$${}_0^C D^\alpha X_A(t) = A + t^{1-\alpha} E_{2,2-\alpha}(-t^2)$$

Since $g(t) = t^{1-\alpha} E_{2,2-\alpha}(-t^2) \rightarrow 0$ as $t \rightarrow \infty$, it is bounded below.

Choosing $A > -\inf_{t \geq 0} g(t)$ guarantees ${}_0^C D^\alpha X_A(t) > 0$ globally.

2. Oscillatory Behavior:

$$X'_A(t) = \frac{A}{\Gamma(\alpha)} t^{\alpha-1} + \cos(t)$$

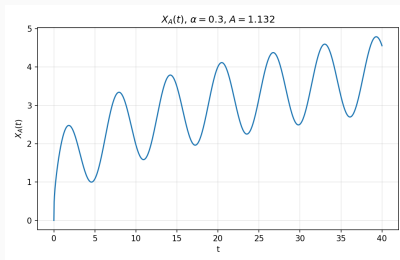
Since $t^{\alpha-1} \rightarrow 0$, $X'_A(t)$ endlessly alternates between -1 and 1 .

Conclusion: $X_A(t)$ oscillates indefinitely despite ${}_0^C D^\alpha X_A(t) > 0$.

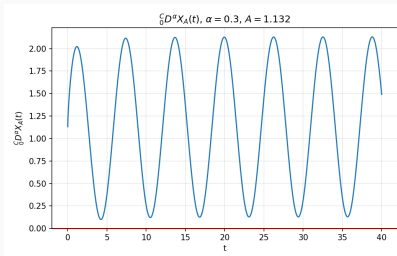
Loss of Monotonicity Counterexample: Numerical Illustration

Example: $\alpha = 0.3$, $A \approx 1.13$ ($\inf g \approx -1.03$)

$$X_A(t)$$



$${}_0^C D^\alpha X_A(t)$$



X_A is not monotone, yet ${}_0^C D^\alpha X_A > 0$ for all t .

Fundamental theorems for fractional calculus

Let $\alpha \in (0, 1)$.

Theorem (Fractional Fundamental Theorem)

1. **In L^p :** If $f \in L^p(t_0, T)$, $p \geq 1$, then

$${}^C_{t_0}D^\alpha ({}^{\text{RL}}_{t_0}I^\alpha f) = f \quad \text{a.e. on } [t_0, T]$$

2. **In continuous functions:** If $f \in \mathcal{C}([t_0, T])$, then

$${}^C_{t_0}D^\alpha ({}^{\text{RL}}_{t_0}I^\alpha f) = f \quad \text{on } [t_0, T]$$

3. **Recovery formula:** If $f \in \mathcal{C}([t_0, T])$ and ${}^C_{t_0}D^\alpha f \in \mathcal{C}([t_0, T])$, then

$${}^{\text{RL}}_{t_0}I^\alpha ({}^C_{t_0}D^\alpha f) = f - f(t_0)$$

Continuity of fractional derivative: Vainikko's characterization (1/2)

Theorem

For an $\alpha \in (0, 1)$ and a function $f \in \mathcal{C}([t_0, T])$, the following conditions are equivalent:

(i) ${}^C_{t_0}D^\alpha f \in \mathcal{C}([t_0, T])$.

(ii) The limit $\lim_{t \rightarrow t_0} (t - t_0)^{-\alpha} (f(t) - f(t_0))$ exists, the improper Riemann integral

$$\int_{t_0}^t (t - s)^{-\alpha-1} (f(t) - f(s)) \, ds$$

converges for every $t \in (t_0, T]$ and defines a continuous function on $(t_0, T]$ that has a continuous extension at t_0 . Moreover, there exists a function $W \in L^1(t_0, T)$ such that

$$\left| \int_{t_0}^{\theta t} (t - s)^{-\alpha-1} (f(t) - f(s)) \, ds \right| \leq W(t).$$

Continuity of fractional derivative: Vainikko's characterization (2/2)

Theorem (continued)

(iii) The limit $\lim_{t \rightarrow 0} (t - t_0)^{-\alpha} (f(t) - f(t_0))$ exists, and

$$\sup_{t_0 < t \leq T} \left| \int_t^{\theta t} (t - s)^{-\alpha-1} (f(t) - f(s)) ds \right| \rightarrow 0 \quad \text{as } \theta \uparrow 1.$$

Moreover, for all $t \in (t_0, T]$,

$$\begin{aligned} {}^C_{t_0} D^\alpha f(t) = & \frac{1}{\Gamma(1-\alpha)} \left((t - t_0)^{-\alpha} (f(t) - f(t_0)) \right. \\ & \left. + \alpha \int_{t_0}^t (t - s)^{-\alpha-1} (f(t) - f(s)) ds \right). \end{aligned}$$

Ref.: G. Vainikko (2016), *Which functions are fractionally differentiable?*, Z. Anal. Anwend., 35(4), 465–487.

as $0 < h \rightarrow 0$. Indeed, fix an arbitrary small $\varepsilon > 0$. Since $v_0 \in \mathcal{H}_0^\alpha(0, T]$, there is a $\delta > 0$ such that

$$|v_0(t) - v_0(s)| \leq \varepsilon(t-s)^\alpha \quad \text{for } 0 \leq t-s \leq \delta. \quad (22)$$

Hence, for $0 < h \leq \delta$ we obtain $\int_{t-h}^t \left| \frac{(t+h-s)^{-\alpha} - (t-s)^{-\alpha}}{h} \right| |v_0(t) - v_0(s)| ds \leq \varepsilon \int_{t-h}^t \frac{(t-s)^{-\alpha} - (t+h-s)^{-\alpha}}{h} (t-s)^\alpha ds = \varepsilon \frac{1}{h} \int_{t-h}^t \left(1 - \left(\frac{t-s}{t+h-s}\right)^\alpha\right) ds \leq \varepsilon \frac{1}{h} \int_{t-h}^t ds = \varepsilon$ proving (20). To prove (21), represent

$$\frac{(t+h-s)^{-\alpha} - (t-s)^{-\alpha}}{h} = -\alpha(t+h'-s)^{-\alpha-1}, \quad \text{with some } h' \in (0, h),$$

and using (22) and (17) estimate

$$\begin{aligned} & \int_0^{t-h} \left| \frac{(t+h-s)^{-\alpha} - (t-s)^{-\alpha}}{h} + \alpha(t-s)^{-\alpha-1} \right| |v_0(t) - v_0(s)| ds \\ & \leq \varepsilon \alpha \int_0^{t-h} ((t-s)^{-\alpha-1} - (t+h'-s)^{-\alpha-1})(t-s)^\alpha ds \\ & \leq \varepsilon \alpha \int_0^{t-h} ((t-s)^{-\alpha-1} - (t+h-s)^{-\alpha-1})(t-s)^\alpha ds \\ & \leq c_\alpha^+ \alpha \varepsilon \quad \text{for } 0 < h < \delta. \end{aligned}$$

Some partial results concerning Vainikko's Theorem

- The Vainikko's characterization theorem is true if we assume that $f \in \mathcal{AC}[t_0, T]$.
- The Vainikko's characterization theorem holds for functions of the subspace $\mathcal{CW}^1([0, T])$, where

$$\mathcal{CW}^1([t_0, T]) = \{f \in \mathcal{C}([t_0, T]), f' \in \mathcal{C}((t_0, T)) \cap L^1(t_0, T)\}.$$

A chain rule inequality for the Caputo operator

Theorem

Let $\alpha \in (0, 1)$ and $f \in \mathcal{C}([t_0, T])$ such that ${}^C_{t_0}D^\alpha f \in \mathcal{C}([t_0, T])$.

If $V \in \mathcal{C}^1(\mathbb{R}^d, \mathbb{R})$ is a convex function, then the fractional derivative

${}^C_{t_0}D^\alpha(V \circ f) \in \mathcal{C}([t_0, T])$, and the following estimate holds:

$${}^C_{t_0}D^\alpha(V \circ f)(t) \leq \nabla V(f(t)) \cdot {}^C_{t_0}D^\alpha f(t) \quad \forall t \in (t_0, T].$$

Ref.: Tuan, H. T., & Trinh, H. (2018). Stability of fractional-order nonlinear systems by Lyapunov direct method. IET Control Theory & Applications, 12(17), 2417-2422.

Open Problems and Future Directions

From Power-Law to Sonine Kernels

Classical Foundation: The Power-Law Pair

Classical Riemann-Liouville fractional calculus is built upon the singular power-law kernel:

$$k_{\alpha}(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \quad (\alpha \in (0, 1))$$

and its associated dual kernel $k_{1-\alpha}(t) = \frac{t^{-\alpha}}{\Gamma(1-\alpha)}$, which satisfy the fundamental convolution identity:

$$(k_{\alpha} * k_{1-\alpha})(t) = \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\alpha} ds = 1 \quad \forall t > 0.$$

Core Idea of Generalized Fractional Calculus

Replace the specific power-law pair $(k_{\alpha}, k_{1-\alpha})$ with a general pair of functions (k, κ) that satisfies the **exact same convolution identity**.

Definition and Examples of Sonine Kernels

Definition (Sonine Condition (Sonine, 1884))

Two functions $k, \kappa \in L^1_{\text{loc}}(0, \infty)$ form a pair of **Sonine kernels** if:

$$(k * \kappa)(t) = \int_0^t k(t-s)\kappa(s) ds = 1 \quad \text{for almost all } t > 0.$$

$\kappa(t)$ is called the **associated (dual) Sonine kernel** of $k(t)$.

Examples of Sonine Pairs (k, κ)

1. Multi-Term / Regularized Kernels:

$$k(t) = t^{-\alpha} E_{\beta, 1-\alpha}(-\lambda t^{\beta}) \quad \implies \quad \kappa(t) = t^{\alpha-1} + \lambda t^{\alpha+\beta-1}$$

2. Distributed-Order Kernels: $k(t) = \int_0^1 \frac{t^{\alpha-1}}{\Gamma(\alpha)} \mu(\alpha) d\alpha$, with dual $\kappa(t)$ determined in the Laplace domain by $\tilde{\kappa}(s) = \frac{1}{s\tilde{k}(s)}$.

General Fractional Integrals and Derivatives

Let (k, κ) be a given pair of Sonine kernels.

Definition (General Fractional Operators (Kochubei, 2011))

- **General Fractional Integral:**

$${}_{t_0}\mathbb{I}_k f(t) = (k * f)(t) = \int_{t_0}^t k(t-s)f(s) ds$$

- **Riemann-Liouville Type:**

$${}_{t_0}^{\text{RL}}\mathbb{D}_\kappa f(t) = \frac{d}{dt}\mathbb{I}_\kappa f(t) = \frac{d}{dt} \int_{t_0}^t \kappa(t-s)f(s) ds$$

- **Caputo Type:**

$${}_{t_0}^{\text{C}}\mathbb{D}_\kappa f = {}_{t_0}^{\text{RL}}\mathbb{D}_\kappa [f - f(t_0)]$$

Ref.: Kochubei, A. N. (2011). General fractional calculus, evolution equations, and renewal processes. *Integral Equations and Operator Theory*, 71(4), 583-600.

- **Vainikko-type Regularity:** Extending Vainikko's necessary and sufficient condition for continuous fractional differentiability to general Sonine operators ${}^C_{t_0}D_{\kappa}$.
- **Lyapunov Convexity Inequalities:** Establishing non-local chain rule inequalities of ${}^C_{t_0}D_{\kappa}(V \circ f)$ for convex non-smooth candidate Lyapunov functions V under generalized Sonine memory kernels.

Open Research Directions in Fractional Calculus

- **Väinikko's Conjecture** : Let $0 < \alpha < 1$ and $1 < p \leq \infty$. A function $v \in L^p(t_0, T)$ has a Caputo derivative ${}^C_{t_0}D^\alpha v \in L^p(t_0, T)$ if and only if:
 - (i) $(t - t_0)^{-\alpha} v(t) \in L^p(t_0, T)$,
 - (ii) $w(t) := \int_{t_0}^t (t - s)^{-\alpha-1} (v(t) - v(s)) ds$ is well-defined a.e. in (t_0, T) and $w \in L^p(t_0, T)$.
- **Current Status**: We constructed an explicit counterexample disproving the equivalence.
- **Open Challenge**: Formulate and prove a correct characterization for $v \in L^p(t_0, T)$ to possess an integrable Caputo fractional derivative ${}^C_{t_0}D^\alpha v \in L^p(t_0, T)$.

Ref: Lamzib, S., Lahrouz, A., & Zeghal, A. (2026). Absolute Marchaud integrability is not necessary for membership in Riemann–Liouville L^p images. arXiv preprint arXiv:2610.06867.

Open Research Directions in Fractional Calculus

- **Fractional LaSalle's Invariance Principle** Consider the fractional-order initial value problem:

$${}^C_{t_0}D^\alpha X(t) = F(t, X(t)), \quad t \in (t_0, T], \quad X(t_0) = X_0 \in \Omega. \quad (5)$$

- **False Lemma:** Let $D \subset \mathbb{R}^n$ be a compact positively invariant set for (5). Assume $V \in C^1(D, \mathbb{R})$ satisfies $D^\alpha V|_{(5)} \leq 0$. Let M be the largest invariant set contained in $E = \{x \in D : D^\alpha V|_{(5)} = 0\}$. Then every solution $x(t)$ originating in D satisfies $x(t) \rightarrow M$ as $t \rightarrow \infty$.

Critical Note

This formulation is **false** and ill-posed: the fractional derivative ${}^C_{t_0}D^\alpha V(X(t))$ depends explicitly on time rather than purely on the state X , making the definition of E invalid. Furthermore, the proof relies on classical monotonicity arguments that fail under non-local memory effects. Despite these fundamental flaws, this result has accumulated 376 citations in the literature.

Ref: Huo, J., Zhao, H., & Zhu, L. (2015). The effect of vaccines on backward bifurcation in a fractional order HIV model. *Nonlinear Analysis: Real World Applications*, 26, 289–305.

Thank you for your attention!

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